Finite-time vortex singularity and Kolmogorov spectrum in a symmetric three-dimensional spiral model

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(Received 5 October 1994; revised manuscript received 30 May 1995)

A recent analytical model of three-dimensional Euler flows [Phys. Rev. Lett. **69**, 2196 (1992)] which exhibits a finite-time vortex singularity is developed further. The initial state is symmetric and contains a velocity null (stagnation point) which is collinear with two vorticity nulls. Under some assumptions, it is shown by asymptotic analysis of the Euler equation that the vorticity blows up at the stagnation point as inverse time in a locally self-similar manner. The spatial structure of the inviscid flow in the vicinity of the singularity involves disparate small scales. The effect of a small but finite viscosity is shown to arrest the formation of the singularity. The presence of spiral structure in the initial conditions leads naturally to the model developed by Lundgren [Phys. Fluids **25**, 2193 (1982)] in which the gradual tightening of spirals by differential rotation provides a mechanism for transfer of energy to small spatial scales. It is shown by asymptotic analysis of the Navier-Stokes equation, that a time-average over the lifetime of the spiral vortex in the present model yields the Kolmogorov spectrum.

PACS number(s): 47.27.Cn

I. INTRODUCTION

A fundamental problem in fluid dynamics is the derivation of the Kolmogorov spectrum for incompressible turbulent flows from the Navier-Stokes equation. The Kolmogorov scaling law [1], originally derived by dimensional analysis, has been validated by several experiments and numerical simulations, but its theoretical derivation from the underlying dynamical equations has remained a challenge for over 50 years.

It is widely believed that a first step in the development of a dynamical theory of turbulence should be the identification of a mechanism by which energy in the large spatial scales can be transferred to the small scales. This has motivated the search for finite-time vortex singularities in three-dimensional Euler flows, since in two-dimensional flows that tend to zero at infinity and evolve from smooth initial conditions, the formation of a finite-time vortex singularity is forbidden [2–4].

Theoretical models that may yield finite-time vortex singularities in three-dimensional Euler flows have been the subject of many investigations, most of which are numerical [5–9]. Until 1990, the results were inconclusive despite the sophistication of the numerical methods employed. Recently, finite-time singularities have been reported in axisymmetric flows with swirl [10,11], but doubts have been raised that the growth of vorticity observed in these studies may be exponential [12]. A related problem has been investigated by Childress [13], who has interpreted the occurrence of a finite-time singularity in a nearly two-dimensional flow as a signature of the loss of near-two-dimensionality. E and Shu [14] present a numerical study of two-dimensional Boussinesq convection (also studied earlier by Grabowski and Clark [15]) and report exponential growth of vorticity in those regions where Pumir and Siggia [11] find a finite-time singularity. Despite the depth of effort that has gone into these investigations, it cannot be said that the issue of vorticity intensification in axisymmetric flows with swirl has been settled beyond doubt. E and Shu find that the growth of vorticity is much more intense on the side of a rising bubble than the cap where the growth of vorticity is found to be exponential. Caflisch [16] has presented a different viewpoint on this problem and has demonstrated the development of finite-time vortex singularities in a model of complex-valued axisymmetric flows with swirl. However, the singularities found by Caflisch do not occur where they do in Refs. [10] and [11], but at the centers of the rolls. Though Caflisch's results are mathematically interesting, there are questions regarding the relevance of his findings to the real dynamics of the Euler equation.

In a way, the controversy surrounding the problem of axisymmetric flows with swirl encapsulates the difficulties involved in the search for finite-time vortex singularities. The question is not only *how rapidly in time* vorticity grows, but also *where* the singular growth of vorticity should occur. The first question involves dynamics, whereas the second involves geometry. In attempting to answer the second question, we adopt the point of view that vortex singularities will tend to occur near separatrices. We arrive at this point of view, prompted by insights on the analogous problem of current singularity formation in magnetohydrodynamics (MHD). Though there are profound differences between the dynamics of magnetic fields in plasmas and velocity fields in Euler flows, there are some geometrical similarities. We discuss below some of the similarities and differences.

Magnetic fields $\mathbf{B}$ are divergence-free, as are incompressible velocity fields $\mathbf{v}$. The analogy between $\mathbf{B}$ and $\mathbf{v}$ is most obvious when we consider steady solutions of the Euler equation

$$\mathbf{v}\times\boldsymbol{\omega}=\nabla h \; , \quad \nabla\times\mathbf{v}=\boldsymbol{\omega} \; , \qquad (1)$$

where $\boldsymbol{\omega}$ is the vorticity and $h=p/\rho+v^2/2$ is the Ber-

1063-651X/95/52(5)/5110(14)/$06.00

noulli function for a fluid of pressure p and density ρ. Equation (1) is analogous to the magnetostatic condition [17]

$$\mathbf{J}\times\mathbf{B}=\nabla p, \quad \nabla\times\mathbf{B}=\mathbf{J} , \tag{2}$$

where $\mathbf{J}$ is the electrical current density and p is the fluid pressure. A comparison of (1) and (2) suggests the analogy $\mathbf{B}\leftrightarrow\mathbf{v}$, $\mathbf{J}\leftrightarrow\boldsymbol{\omega}$, $p\leftrightarrow h_0-h$ (where h_0 is a constant). It is this analogy that has provided the basis for certain deductions regarding the properties of steady three-dimensional Euler flows in Refs. [18–22]. In Ref. [22], solutions of (1) that are topologically toroidal are considered and it is demonstrated that δ-function vortex singularities can occur at the so-called rational surfaces of a nearly integrable velocity field. These rational surfaces, on which the streamlines close on themselves, are the source of separatrices in a torus. The vortex singularities at the rational surfaces are exactly where current singularities occur in the toroidal solutions of a nearly integrable magnetic field [21].

This analogy between $\mathbf{B}$ and $\mathbf{v}$ in steady state breaks down when we consider dynamics. In an ideal plasma, the magnetic field $\mathbf{B}$ obeys the induction equation

$$\frac{\partial\mathbf{B}}{\partial t}-\nabla\times(\mathbf{v}\times\mathbf{B})=\mathbf{0} , \tag{3}$$

whereas the vorticity $\boldsymbol{\omega}$ (and *not* the velocity) obeys the (analogous) Euler equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t}-\nabla\times(\mathbf{v}\times\boldsymbol{\omega})=\mathbf{0} . \tag{4}$$

In other words, for dynamical evolution, the appropriate analogy is $\mathbf{B}\leftrightarrow\boldsymbol{\omega}$ (to be contrasted with $\mathbf{J}\leftrightarrow\boldsymbol{\omega}$ in steady state). The ramifications of this dynamical analogy have been partially explored in Refs. [19–22] and has prompted Greene [23] to suggest "...that an essential aspect of turbulence is that it is a dynamo for vorticity amplification... ." It is because of this analogy that mathematical methods used in MHD have proved to be valuable in obtaining new stability results in Euler flows with stagnation points [24,25]. The analogy suggests that since in MHD magnetic-field nulls are a possible source of separatrices [26,27] where singular currents tend to grow, the neighborhood of vorticity and velocity nulls are possible sites for singularity formation in three-dimensional Euler flows.

Recently, we have proposed an analytical model [28] of three-dimensional Euler flows containing nulls. The initial state of this model (discussed in Sec. II) is symmetric and has a stagnation point at the origin, flanked by two vorticity nulls. The straight line joining the two vorticity nulls intersects the stagnation point at the origin, which is preserved for all times by the flow. In Ref. [28] we have attempted to show that this model yields a finite-time singularity by means of a multiple-scale analysis under some strong assumptions regarding the form of the solutions. In Sec. III we give a modified version of this analysis that makes the derivation of the singularity more transparent. The modified analysis, which is carried out to higher order, demonstrates that there are two disparate spatial scales in the neighborhood of the stagnation point. As in Ref. [28], we find that the vorticity blows up as $(t_c-t)^{-1}$ at the stagnation point in a locally self-similar manner within a collapsing inner region. While the velocity remains bounded in the inner region, it blows up as $(t_c-t)^{-1/2}$ in a surrounding region where the flow violates the self-similar scaling of the inner region.

The results obtained in Sec. III by the perturbative analysis motivate a more general treatment of stagnation point flows (Sec. IV) in which the pressure is calculated self-consistently by a (nonlocal) integral relation involving the velocity. This generalized treatment enables us to consider a wider class of initial conditions (of finite energy) than that considered in Sec. III. The results of Sec. IV support the results obtained in Sec. III and furthermore suggest that there are broadly two types of vortex singularities that can be realized with initial conditions of high symmetry.

In Sec. V, we study the models discussed in Secs. III and IV in the complex spatial domain, following the recent treatment of Tanveer and Speziale [29]. In this approach, the Euler equation is continued into the complex unphysical domain. Since a class of smooth and bounded initial conditions in the real physical domain can be singular when analytically continued into the complex unphysical domain, the question of finite-time singularities in the real physical domain can be reduced to the question of whether the singularities in the complex domain reach the real domain finite time. The analytical framework developed in Ref. [29] is based on a crucial set of assumptions that have not been shown to hold in general. Whether these assumptions hold for the symmetric dynamics considered in this paper also remains unproved. However, the approach of Tanveer and Speziale does reproduce the two disparate small scales found by the asymptotic analysis of Sec. III.

We discuss in Sec. VI that the analytical results of Secs. III–V do not violate the rigorous constraints due to Constantin and Fefferman [30,31]. However, we point out that the relevant theorems in Refs. [30] and [31] are proved under a specific assumption that appears to be violated by self-similar flows and hence it is questionable whether these theorems are directly applicable to our model.

One of the interesting features of the smooth initial state in Ref. [28] is the presence of spiral structures. Though the spiral structure does not play a significant role in the inviscid evolution of the flow, its occurrence is reminiscent of the strained vortex model of Lundgren [32–34]. In Lundgren's model, nonaxisymmetric coherent vortices with spiral structures interact with each other only through the mediation of an axisymmetric, background straining flow. By means of detailed asymptotic analysis, Lundgren demonstrates that the Kolmogorov spectrum then follows from the Navier-Stokes equation.

Lundgren's discovery, and the natural presence of spirals in our model, motivates us to investigate the effect of viscosity (Sec. VII). It is shown that the presence of viscosity arrests the formation of the finite-time vortex singularity in our model. As the vorticity intensifies at the velocity null, the tightening of the spirals in our initial state provides a mechanism for transfer of energy from the large to the small spatial scales. An asymptotic analysis based on the Navier-Stokes equation leads us quite naturally to Lundgren's considerations and we too obtain the Kolmogorov spectrum. Viewed in its totality, the present model possesses some features that enhance Lundgren's conception. In particular, the model describes the dynamics of a three-dimensional (but symmetric) flow that evolves from smooth initial conditions and tends to a finite-time singularity (which Lundgren's model does not). Subsequently, viscosity thwarts the formation of the singularity and yields the Kolmogorov spectrum (as Lundgren's model does). Thus the present model can claim to capture some features of decaying hydrodynamic turbulence.

II. INITIAL CONDITIONS

At $t=0$, we consider a symmetric flow $\mathbf{u}$ of the form

$$u_x=f(y)\ , \tag{5a}$$

$$u_y=f(z)\ , \tag{5b}$$

$$u_z=f(x)\ . \tag{5c}$$

For specificity, we choose

$$f(x)=u_0\frac{x}{a_0}\exp\left[-\frac{\varepsilon x^2}{a_0^2}\right]\ , \tag{6}$$

where u_0 and a_0 are positive constants and ε is a small and positive parameter (i.e., $\varepsilon \ll 1$) that separates the local scale a_0 from the larger scale of the globally extended Gaussian envelope of width $\varepsilon^{-1/2}a_0$. The origin is a stagnation point or a null of the velocity field. Near the origin, we have

$$\mathbf{u}=\mathbf{x}\cdot(\nabla\mathbf{u})_0\ , \tag{7}$$

where the tensor

$$(\nabla\mathbf{u})_0=f'(0)\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \tag{8}$$

has the eigenvalues $\lambda_0=f'(0)$ and $\lambda_\pm=f'(0)\exp(\pm i2\pi/3)$. According to the standard nomenclature [35,36] since $f'(0)=u_0/a_0>0$, this null is of type A_s. Near the null, the eigenvectors for the complex eigenvalues $\lambda_\pm$ lie on the Σ_{A_s} surface, which is the stable manifold, whereas the eigenvector for the real and positive eigenvalue λ_0 lies on the γ_{A_s} line, which is the unstable manifold. The subscript s denotes the spiraling trajectories of the streamlines into the null in the Σ_{A_s} surface.

From the relation $\boldsymbol{\omega}=\nabla\times\mathbf{u}$, we obtain

$$\omega_x=-f'(z)=-\frac{u_0}{a_0}\left[1-\frac{2\varepsilon z^2}{a_0^2}\right]\exp\left[-\frac{\varepsilon z^2}{a_0^2}\right]\ , \tag{9a}$$

$$\omega_y=-f'(x)=-\frac{u_0}{a_0}\left[1-\frac{2\varepsilon x^2}{a_0^2}\right]\exp\left[-\frac{\varepsilon x^2}{a_0^2}\right]\ , \tag{9b}$$

$$\omega_z=-f'(y)=-\frac{u_0}{a_0}\left[1-\frac{2\varepsilon y^2}{a_0^2}\right]\exp\left[-\frac{\varepsilon y^2}{a_0^2}\right]\ . \tag{9c}$$

At the origin, $\omega_x=\omega_y=\omega_z=-u_0/a_0$. It follows from Eqs. (9) that the vorticity has two nulls at $x=y=z=\pm a_0/\sqrt{2\varepsilon}\equiv a_\pm$. The vorticity null $x=y=z=a_+$ is of type A_s, whereas the null $x=y=z=a_-$ is of type B_s. A null of type B_s is characterized by one real, negative eigenvalue and two complex eigenvalues. The eigenvector for the real, negative eigenvalue lies on the γ_{B_s} curve, which is a stable manifold. The eigenvectors for the two complex eigenvalues lie on a two-dimensional plane that coincides with the Σ_{B_s} surface, which is an unstable manifold.

Expanding Eqs. (5), (6), and (9) in Taylor series, we obtain

$$u_x=s_0y\left[1-\frac{\varepsilon y^2}{a_0^2}\right]+O(\varepsilon^2)\ . \tag{10a}$$

$$u_y=s_0z\left[1-\frac{\varepsilon z^2}{a_0^2}\right]+O(\varepsilon^2)\ , \tag{10b}$$

$$u_z=s_0x\left[1-\frac{\varepsilon x^2}{a_0^2}\right]+O(\varepsilon^2)\ , \tag{10c}$$

whence

$$\omega_x=-s_0\left[1-\frac{3\varepsilon z^2}{a_0^2}\right]+O(\varepsilon^2)\ , \tag{11a}$$

$$\omega_y=-s_0\left[1-\frac{3\varepsilon x^2}{a_0^2}\right]+O(\varepsilon^2)\ , \tag{11b}$$

$$\omega_z=-s_0\left[1-\frac{3\varepsilon y^2}{a_0^2}\right]+O(\varepsilon^2)\ , \tag{11c}$$

where $s_0=u_0/a_0$. The symmetry of the initial conditions singles out the $x=y=z$ line as a natural axis. We introduce a new coordinate system (x',y',z'), where $\hat{\mathbf{z}}'=(1/\sqrt{3})(1,1,1)$ and $\hat{\mathbf{x}}',\hat{\mathbf{y}}'$ are two mutually orthogonal unit vectors in the plane normal to $\hat{\mathbf{z}}'$, with $\hat{\mathbf{x}}'=(1/\sqrt{6})(-1,-1,2)$ and $\hat{\mathbf{y}}'=(1/\sqrt{2})(1,-1,0)$. The initial flow (10) can be written as

$$\mathbf{u}=\begin{bmatrix}u_{x'}\\u_{y'}\\u_{z'}\end{bmatrix}=s_0\begin{bmatrix}-1/2&\sqrt{3}/2&0\\-\sqrt{3}/2&-1/2&0\\0&0&1\end{bmatrix}\begin{bmatrix}x'\\y'\\z'\end{bmatrix}+O(\varepsilon)\ . \tag{12}$$

Defining $x'=r'\cos(\theta'+\pi/6)$, $y'=r'\sin(\theta'+\pi/6)$, Eq. (12) can be transformed to

$$\mathbf{u}=\begin{bmatrix}u_r\\u_\theta\\u_z\end{bmatrix}=s_0\begin{bmatrix}-r/2\\-(\sqrt{3}/2)r\\z\end{bmatrix}+O(\varepsilon)\ , \tag{13}$$

where we have dropped the primes for notational convenience. As noted in Ref. [28], Eq. (13) is axisymmetric to leading order. The departure from axisymmetry and the spiral structure is manifest at higher order. The initial vorticity is given by the expansion

$$\boldsymbol{\omega}=-\sqrt{3}s_0\left[\begin{bmatrix}0\\0\\1\end{bmatrix}+\frac{\varepsilon z}{a_0^2}\begin{bmatrix}r\\-\sqrt{3}r\\-z\end{bmatrix}-\frac{\varepsilon r^2}{\sqrt{2}a_0^2}\begin{bmatrix}\sin 3\theta\\ \cos 3\theta\\ \sqrt{2}\end{bmatrix}\right]+O(\varepsilon^2)\ . \tag{14}$$

In what follows, we shall investigate the time evolution of the initial state described above according to the Euler equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\boldsymbol{\omega}=\boldsymbol{\omega}\cdot\nabla\mathbf{u}\ . \tag{15}$$

Here we have done away with the so-called background flow considered in Ref. [28], which is inessential for singularity formation.

III. PERTURBATIVE SOLUTIONS

The localized flow $\mathbf{u}$ obeys the symmetry relations

$$u_x(x,y,z)=u_y(z,x,y)=u_z(y,z,x)\ . \tag{16}$$

Since these symmetry relations are assumed to hold at $t=0$, they hold for all times [37]. We consider solutions of the form

$$u_x(\mathbf{x},t)=s_2(\mathbf{x},t)y\exp\left[-\frac{y^2}{a_2^2(\mathbf{x},t)}\right], \tag{17a}$$

$$u_y(\mathbf{x},t)=s_3(\mathbf{x},t)z\exp\left[-\frac{z^2}{a_3^2(\mathbf{x},t)}\right], \tag{17b}$$

$$u_z(\mathbf{x},t)=s_1(\mathbf{x},t)x\exp\left[-\frac{x^2}{a_1^2(\mathbf{x},t)}\right]. \tag{17c}$$

The six functions on the right-hand side of (17) are constrained by the incompressibility condition and the (two) symmetry relations (16). Exploiting the presence of the small parameter ε in the initial conditions, we seek solutions of the form

$$a_i=a(t)+\varepsilon\tilde{a}_i(\mathbf{x},t)\ , \tag{18a}$$

$$s_i=s(t)+\varepsilon\tilde{s}_i(\mathbf{x},t)\ , \tag{18b}$$

where $a(0)=a_0$ and $s(0)=s_0$. The form of Eq. (18) is based on the assumption that the solutions are locally self-similar. The solutions are not globally self-similar; the space dependence of the higher-order terms in Eq. (18) preclude such a possibility.

For the initial conditions discussed above, we develop solutions of the Euler equation by a formal perturbation expansion. We write

$$u_r=u_r^{(0)}+\varepsilon u_r^{(1)}+O(\varepsilon^2)\ , \tag{19a}$$

$$u_\theta=u_\theta^{(0)}+\varepsilon u_\theta^{(1)}+O(\varepsilon^2)\ , \tag{19b}$$

$$u_z=u_z^{(0)}+\varepsilon u_z^{(1)}+O(\varepsilon^2)\ , \tag{19c}$$

and

$$\omega_r=\omega_r^{(0)}+\varepsilon\omega_r^{(1)}+O(\varepsilon^2)\ , \tag{20a}$$

$$\omega_\theta=\omega_\theta^{(0)}+\varepsilon\omega_\theta^{(1)}+O(\varepsilon^2)\ , \tag{20b}$$

$$\omega_z=\omega_z^{(0)}+\varepsilon\omega_z^{(1)}+O(\varepsilon^2)\ . \tag{20c}$$

Using the symmetry relation (16) and the assumption of local self-similarity, the leading-order solution for the velocity can be written in the form

$$\mathbf{u}^{(0)}=\begin{bmatrix}u_r^{(0)}\\u_\theta^{(0)}\\u_z^{(0)}\end{bmatrix}=s(t)\begin{bmatrix}-r/2\\-(\sqrt{3}/2)r\\z\end{bmatrix}\ . \tag{21}$$

Then the z component of the leading-order Euler equation (15) for $\omega_z^{(0)}=-\sqrt{3}s(t)$ yields

$$\frac{\partial\omega_z^{(0)}}{\partial t}=s\omega_z^{(0)} \tag{22}$$

or

$$\frac{\partial s}{\partial t}=s^2\ . \tag{23}$$

Equation (23) has the exact solution

$$s=\frac{1}{t_c-t}\ , \tag{24}$$

which yields a finite-time singularity at $t_c\equiv 1/s_0=a_0/u_0$. The singularity occurs in a locally self-similar manner in the neighborhood of the velocity null. For all times prior to the blowup of vorticity, there exists a small region surrounding the velocity null where the solution is invariant under the scaling $\mathbf{x}\rightarrow c\mathbf{x}$, $\mathbf{u}\rightarrow\mathbf{u}, t\rightarrow ct$, where c is a constant. In this (shrinking) region, the solutions $u_x=y/(t_c-t)$, $u_y=z/(t_c-t)$, and $u_z=x/(t_c-t)$, which yield $\omega_x=\omega_y=\omega_z=-(t_c-t)^{-1}$, satisfy the Euler equation (15) exactly.

It is useful to calculate the pressure Hessian $\pi_{ij}\equiv\partial^2p/\partial x_i\partial x_j$ for the leading-order solution. We obtain

$$\pi_{ij}^{(0)}=-\begin{bmatrix}0&s^2&s^2\\s^2&0&s^2\\s^2&s^2&0\end{bmatrix}\ . \tag{25}$$

Whereas the diagonal elements, which represents the local contributions to the pressure Hessian, are zero, the off-diagonal elements, which represent the nonlocal contributions, are nonzero (and eventually blow up). Hence the present model cannot be described by the so-called "restricted Euler system" of Leorat [38], Viellefosse [39], and Cantwell [40], who assume that

$\pi_{ij}=(\partial^2 p/\partial x_k \partial x_k)(\delta_{ij}/3)$.

We now consider the first-order solutions. To this end, we note that Eqs. (21) and (24) yield the leading-order Lagrangian equations

$$\frac{dr}{dt}=u_r^{(0)}=-\frac{r}{2(t_c-t)}\ , \tag{26a}$$

$$\frac{d\theta}{dt}=\frac{u_\theta^{(0)}}{r}=-\frac{\sqrt{3}}{2}\frac{1}{t_c-t}\ , \tag{26b}$$

$$\frac{dz}{dt}=u_z^{(0)}=\frac{z}{t_c-t}\ , \tag{26c}$$

which can be integrated to give

$$r=r_0\left[\frac{t_c-t}{t_c}\right]^{1/2}\ , \tag{27a}$$

$$\theta=\theta_0-\frac{\sqrt{3}}{2}\int_0^t dt' s(t')\ , \tag{27b}$$

and

$$z=z_0\frac{t_c}{t_c-t}\ . \tag{27c}$$

respectively.

Writing $\boldsymbol{\omega}=\boldsymbol{\omega}^{(0)}+\varepsilon\boldsymbol{\omega}^{(1)}+\cdots$, we obtain the ($z$ component of the) first-order Euler equation

$$\frac{\partial\omega_z^{(1)}}{\partial t}+(\mathbf{u}\cdot\nabla\omega_z)^{(1)}=(\omega\cdot\nabla u_z)^{(1)}\ . \tag{28}$$

Motivated by Lundgren [32], we introduce the variable transformations

$$\omega_z^{(1)}(r,\theta,z,t)=S(t)\Omega(\xi,\vartheta,v,T)\ , \tag{29a}$$

$$u_z^{(1)}(r,\theta,z,t)=S^{1/2}(t)\Lambda(\xi,\vartheta,v,T)\ , \tag{29b}$$

where

$$S(t)=\exp\left[\int_0^t dt' s(t')\right] \tag{30}$$

is the total stretching in time t and the variable ξ, ϑ, v, and T are defined, respectively, as

$$\xi\equiv S^{1/2}r\ , \tag{31a}$$

$$\vartheta\equiv\theta+\frac{\sqrt{3}}{2}\int_0^t dt' s(t')\ , \tag{31b}$$

$$v\equiv z/S\ , \tag{31c}$$

and

$$T\equiv\int_0^t dt' S(t')\ . \tag{31d}$$

We now calculate the various terms in (28):

$$\left[\frac{\partial\omega_z}{\partial t}\right]^{(1)}=\frac{\partial\omega_z^{(1)}}{\partial t}=\frac{\partial S}{\partial t}\Omega+S\frac{\partial\Omega}{\partial t}=sS\Omega+S\frac{\partial\Omega}{\partial t}=s\omega_z^{(1)}+sS\frac{\partial\Omega}{\partial T}+\frac{s\xi}{2}\left[\frac{\partial\omega_z^{(1)}}{\partial\xi}+\sqrt{3}\frac{\partial\omega_z^{(1)}}{\xi\partial\vartheta}\right]-sv\frac{\partial\omega_z^{(1)}}{\partial v}\ , \tag{32a}$$

$$(\mathbf{u}\cdot\nabla\omega_z)^{(1)}=\mathbf{u}^{(1)}\cdot\nabla\omega_z^{(0)}+\mathbf{u}^{(0)}\cdot\nabla\omega_z^{(1)}\cong\mathbf{u}^{(0)}\cdot\nabla\omega_z^{(1)}=-\frac{s\xi}{2}\left[\frac{\partial\omega_z^{(1)}}{\partial\xi}+\sqrt{3}\frac{\partial\omega_z^{(1)}}{\partial\vartheta}\right]+sv\frac{\partial\omega_z^{(1)}}{\partial v}\ , \tag{32b}$$

$$(\omega\cdot\nabla u_z)^{(1)}=\omega^{(1)}\cdot\nabla u_z^{(0)}+\omega^{(0)}\cdot\nabla_z^{(1)}=s\omega_z^{(1)}-\sqrt{3}s\frac{\partial u_z^{(1)}}{\partial z}=s\omega_z^{(1)}-\sqrt{3}sS^{-1/2}\frac{\partial\Lambda}{\partial v}\ . \tag{32c}$$

Using Eqs. (32), the first-order Euler equation (28) can be reduced to

$$\frac{\partial\Omega}{\partial T}=-\sqrt{3}\frac{1}{S^{3/2}}\frac{\partial\Lambda}{\partial v}\ . \tag{33}$$

As $t\to t_c$, the term on the right-hand side of (33) can be neglected. Hence, in this limit

$$\frac{\partial\Omega}{\partial T}=0\ , \tag{34}$$

which implies that

$$\Omega=\Omega(\xi,\vartheta,v)=\Omega_0(\xi_0(\xi,\theta,v),\vartheta_0(\xi,\theta,v),v_0(\xi,\theta,v))\ , \tag{35}$$

where

$$\Omega_0=\omega_z^{(1)}(r,\theta,z,0)=\tilde{\omega}(r,\theta,z)=\sqrt{3}s_0\left[\frac{r^2+z^2}{a_0^2}\right] \tag{36}$$

is the initial condition for the higher-order solution. Using (29a), we can write

$$\omega_z^{(1)}(r,\theta,z;t)\cong S(t)\tilde{\omega}\left[r\left[\frac{t_c}{t_c-t}\right]^{1/2},\theta+\frac{\sqrt{3}}{2}\int_0^t dt' s(t'),z\left[\frac{t_c-t}{t_c}\right]\right]$$

$$=\frac{\sqrt{3}s_0}{a_0^2}\exp\left[\int_0^t s(t')dt'\right]\left[r^2\left[\frac{t_c}{t_c-t}\right]+z^2\left[\frac{t_c-t}{t_c}\right]^2\right]\ . \tag{37}$$

If follows by inspection of Eq. (37) that the first-order solution has a small radial scale, collapsing as $(t_c-t)^{1/2}$, while it stretches axially along z. The collapsing small radial scale of the first-order solution is thus different from the small scale of the leading-order solution, collapsing as (t_c-t).

The structure of the small scale obtained above can be understood simply by invoking the law of conservation of vorticity strength [41]. We consider a "long-thin" cylindrical vortex tube of strength $\omega_z A=\text{const}$, where $A=\pi r^2$. Since $\omega_z \propto (t_c-t)^{-1}$, it follows that $r \propto (t_c-t)^{1/2}$. While the velocity is bounded in the inner region, which collapses as (t_c-t), the Lagrangian equations (26) imply that the velocity must blow up on the "larger" small scale, which collapses as $(t_c-t)^{1/2}$.

The analysis given above is perturbative and we have not obtained closed-form solutions. The calculation has been essentially carried through the first two orders, which make it apparent that more than one small spatial scale is involved in this inviscid problem. It is possible to extend the calculation, in principle, to higher orders, but the equations are complicated and do not seem amenable to an analytical solution.

A pertinent question is whether the singularity obtained in our model is unphysical since the initial flows have infinite total energy. As discussed in Ref. [42], this is indeed a limitation of earlier investigations of two [42] and three-dimensional [43] solutions of the stagnation-point form. We emphasize an important difference between our system of flows and those considered in Refs. [42] and [43]: in our initial conditions, the velocity (and vorticity fields) are bounded everywhere, including points at infinity. This means that the energy density is initially finite everywhere, including points at infinity. Infinite energy is obtained in our initial conditions merely because our system size is infinite, but the finite-time vortex singularity is not an artifact of the infinite system size.

It is well known that the Euler equation is an integro-differential system. (For instance, the velocity must be calculated self-consistently from the vorticity by carrying out an integration over the whole space with suitable boundary conditions). In Ref. [28] a lengthy analysis (which will not be repeated here) is given to match "local" flows of the form (17) to a "global" symmetric flow of general functional form. We assume here that such global solutions do exist (that is, they satisfy the integro-differential system). To further strengthen this conclusion, we give a generalized analysis in the next section that treats the integro-differential system explicitly.

IV. GENERALIZATIONS OF THE MODEL

The model discussed above can be generalized by keeping its basic symmetric features, but allowing for a more general form of the solution. We write

$$\mathbf{u}(\mathbf{x},t)=[u(x,y,z,t),u(y,z,x,t),u(z,x,y,t)] \tag{38a}$$

and assume that in the vicinity of the origin the Taylor expansion of the flow exists and that it only has odd-order terms, i.e.,

$$u(x,y,z,t)=\sum_{l,m,n\geq 0} a_{lmn}(t)x^l y^m z^n , \tag{38b}$$

where $a_{lmn}(t)=0$ if $l+m+n$ is even. [Specifically, it is reasonable to assume that the Taylor series is convergent for $|\mathbf{x}|<\delta(t)$, where δ is finite and we allow $\delta(t\to t_c)\to 0$.] The symmetry relations (38a) ensure that we have a velocity stagnation point at the origin for all time since the Euler equation preserves these relations, if satisfied initially. We also assume that $u(|\mathbf{x}|\to\infty, t=0)\to 0$ sufficiently rapidly that the initial flow has finite energy. Then flows with bounded energy can be constructed by treating the Euler equation as the integro-differential system

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla p , \tag{39a}$$

$$p(\mathbf{x},t)=\frac{1}{4\pi}\int \frac{\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}' , \tag{39b}$$

where $\rho=1$ and the condition $\nabla\cdot\mathbf{u}=0$ is satisfied, once it is so initially. It is easy to see that the pressure p is still finite at large $\mathbf{x}$ even if $\mathbf{u}$ has a locally self-similar singularity, because such a singularity occurs only in a volume that also tends to zero. This ensures that the flow will remain vanishingly small for large $\mathbf{x}$ and thus the total energy will remain finite for all time, since it is so initially.

We consider the general form of the first order expansion of u,

$$u(x,y,z,t)=b(t)y+c(t)z+O(|\mathbf{x}|^3) , \tag{40}$$

which is consistent with the condition $\nabla\cdot\mathbf{u}=0$. The vorticity $\boldsymbol{\omega}=\nabla\times\mathbf{u}$ is then

$$\omega=-\tilde{\omega}(1,1,1)+O(|\mathbf{x}|^2) , \tag{41}$$

where $\tilde{\omega}=b-c$. Also, the dissipation rate per unit mass is given by

$$\varepsilon\equiv\frac{1}{2}\nu\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]^2=3\nu\sigma^2+O(|\mathbf{x}|^2) , \tag{42}$$

where $\sigma=b+c$. In the cylindrical coordinates introduced in Sec. III, Eqs. (40) and (41) can be written, respectively, as

$$\mathbf{u}=\begin{bmatrix}u_r\\u_\theta\\u_z\end{bmatrix}=\begin{bmatrix}-\sigma r/2\\-(\sqrt{3}/2)\tilde{\omega}r\\\sigma z\end{bmatrix}+O(|\mathbf{x}|^3) \tag{43}$$

and

$$\boldsymbol{\omega}=-\sqrt{3}\tilde{\omega}\hat{\mathbf{z}}+O(|\mathbf{x}|^2) . \tag{44}$$

Because of the symmetry of the flow, the pressure term can be expanded in the form

$$p=-d(x^2+y^2+z^2)-e(xy+yz+zx)+O(|\mathbf{x}|^4) . \tag{45}$$

Equation (45) implies that the pressure Hessian π_{ij} has diagonal as well as off-diagonal elements, all of which, in general, can be important in supporting a finite-time singularity. Whereas the diagonal elements, each proportional to d, are determined self-consistently from local properties of the velocity [i.e., they directly enter the differential form of Poisson's equation $\nabla^2 p = -\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$], the off-diagonal elements, each proportional to e, are determined by the global properties of the flow. If we transform to cylindrical coordinates, the pressure tensor becomes diagonal but not isotropic. Substituting Eqs. (43) and (45) into Eq. (39a), we obtain the exact equations

$$d = bc \ , \tag{46a}$$

$$\dot{b} + c^2 = e \ , \tag{46b}$$

$$\dot{c} + b^2 = e \ , \tag{46c}$$

where an overdot indicates the derivative with respect to time. For self-consistency, e must be calculated using Eqs. (39b) and (45):

$$e = -\left.\frac{\partial^2 p}{\partial x \partial y}\right|_{\mathbf{x}=0} = -\int \frac{\partial^2 G(\mathbf{x},t)}{\partial x \partial y}\frac{d^3x}{|\mathbf{x}|} \ , \tag{47}$$

where $G = \nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})/4\pi$. Equations (46b) and (46c) can be rewritten as

$$\dot{\varpi} = \varpi\sigma \ , \tag{48a}$$

$$\dot{\sigma} = 2e - (\varpi^2 + \sigma^2)/2 \ . \tag{48b}$$

From the form of Eqs. (48a) and (48b), it is easy to see that finite-time singularities in ϖ and σ can occur due to the presence of the nonlinear terms for many different functional forms of e. We discuss some interesting examples below.

We consider the case in which the flow is assumed to have the self-similar form

$$\mathbf{u}(\mathbf{x},t) = b[(y,z,x) + a\mathbf{v}_1(\mathbf{x}/a)] + c[(z,x,y) + a\mathbf{v}_2(\mathbf{x}/a)] \ , \tag{49}$$

where a is a function of time only and $\mathbf{v}_1$ and $\mathbf{v}_2$ are two dimensionless functions with no first-order terms in their Taylor expansions. Then it can be shown by dimensional analysis of Eq. (45) that

$$e = c_1 b^2 + c_2 bc + c_3 c^2 \ , \tag{50}$$

where c_1, c_2, c_3 are constants that can be determined once the functional forms of $\mathbf{v}_1$ and $\mathbf{v}_2$ are known. There are numerous choices one can make for c_i and the initial values of b and c that yield singular solutions for Eqs. (46b) and (46c). For example, if we take

$$c_1 = 1, \quad b(0) = 1/t_c, \quad c(0) = 0 \ , \tag{51}$$

then we get

$$b = \varpi = \sigma = 1/(t_c - t) \ , \tag{52}$$

which is the solution obtained in Sec. III.

We now consider slightly more general solutions. Let us assume that the flow is quasi-self-similar, i.e., $\mathbf{u}$ can be approximated by Eq. (49) plus a slowly evolving background flow so that

$$e = c_1 b^2 + c_2 bc + c_3 c^2 + c_4 b + c_5 c + c_6 \ , \tag{53}$$

where c_i are slowly varying functions of time (much before the blowup time t_c). As the solutions evolve from different smooth initial conditions to a self-similar asymptotic state, we find that there are two types of singular solutions:

$$\text{(i)} \quad \sigma = \frac{1}{t_c - t}, \quad \varpi \neq 0$$

$$\text{(ii)} \quad \sigma = \frac{k}{t_c - t}, \quad |\varpi/\sigma| \to 0 \ .$$

This can be seen by substituting $b = k_1/(t_c - t)$ and $c = k_2/(t_c - t)$ into Eq. (50) [or (53)], whereupon using Eq. (46b) [or (46c)], we get

$$(k_1 - k_2)(k_1 + k_2 - 1) = 0 \ . \tag{54}$$

Type (i) and (ii) singularities are realized, respectively, when the second and first factors are set equal to zero.

Type (i) and (ii) singularities can also be obtained by considering a self-similar solution to the Euler equation of the form

$$\mathbf{u}(\mathbf{x},t) = \frac{a(t)}{t_c - t}\mathbf{V}(\mathbf{x}/a), \quad a(t) = \mu(t_c - t)^p \ . \tag{55}$$

The curl of Eq. (39a) yields

$$\nabla\times[(1-p)\mathbf{V}(\mathbf{X}) + (p\mathbf{X} + \mathbf{V})\cdot\nabla\mathbf{V}] = \mathbf{0} \ , \tag{56}$$

where $\mathbf{X} = \mathbf{x}/a$. Let

$$\mathbf{V} = [BY + CZ, BZ + CX, BX + CY] + O(|\mathbf{X}|^2) \ ; \tag{57}$$

we then have

$$(B - C)(B + C - 1) = 0 \ , \tag{58}$$

which is similar to Eq. (54).

In writing Eq. (55), we have assumed the global existence of a divergence-free function $\mathbf{V}(\mathbf{X})$. One way to test this assumption is by rewriting Eq. (55) in the form of an integral equation

$$\mathbf{V}(\mathbf{X}) = -\frac{1}{4\pi(1-p)}\int \frac{\nabla'\times[\{p\mathbf{X}' + \mathbf{V}(\mathbf{X}')\}\cdot\nabla'\mathbf{V}(\mathbf{X}')]\times(\mathbf{X}-\mathbf{X}')}{|\mathbf{X}-\mathbf{X}'|^3}d\mathbf{X} \ , \tag{59}$$

which may be tractable numerically. This is a well-posed problem, but we will not try to solve it here.

The model analyzed in Sec. III is a special case of (i) with $c = 0$ and leads to a finite-time vortex singularity at the origin. At first glance, case (ii) may appear to violate the Beale-Kato-Majda constraint [44] that the maximum vorticity of a flow with an algebraic finite-time singularity has to blow up at least at fast as $(t_c - t)^{-1}$. However, this is not necessarily so. Since both b and c blow up at a rate proportional to $(t_c-t)^{-1}$, it is clear by inspection of (49) that the maximum vorticity due to higher-order terms must also blow up as $(t_c-t)^{-1}$ if $\nabla\times(\mathbf{v}_1+\mathbf{v}_2)\neq 0$, which is generally the case. (A more precise statement is given at the end of Sec. VII.)

Whereas the model considered in Ref. [28] (and Sec. III) is a singular solution of type (i), the singularities obtained in the high-symmetry numerical experiments of Refs. [45] and [46] are examples of type (ii). Consider, for example, the singularity obtained by Kerr in a numerical study of two interacting, antiparallel vortex tubes [45]. Because of the symmetries of Kerr's initial condition, the velocity and vorticity remain zero at the point $\mathbf{x}_c$ where the singularity eventually develops as the two tubes approach each other. The Taylor expansion of the flow about $\mathbf{x}_c$ can be written as

$$\mathbf{v}=[a_1x,a_2y,-(a_1+a_2)z]+O(|\mathbf{x}|^2)\ , \tag{60}$$

where a_1,a_2,a_3 are constants. We note that the $|\mathbf{x}|^2$ terms in Eq. (60) do not affect the analysis given above as long as $\mathbf{v}(\mathbf{x}=0)$ is fixed at zero. Then it follows that in Kerr's geometry a self-similar singular solution should occur with maximum vorticity blowing up as $(t_c-t)^{-1}$.

We conclude with some numerical examples. With e given by (50), we plot in Fig. 1 an example of a type (i) singularity, with $c_1=0.9$, $c_2=-0.9$, $c_3=-0.99$, and the initial conditions $b(0)=1$ and $c(0)=0.5$. In Fig. 2 we plot an example of a type (ii) singularity, with $c_1=10.1$, $c_2=-1.3$, $c_3=-1.45$, and the initial conditions $b(0)=1$ and $c(0)=1.0e-06$.

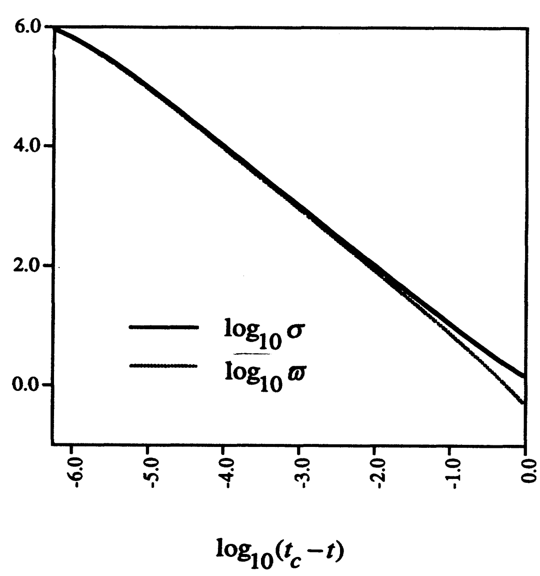

FIG. 1. Example of a singular solution of type (i) where both $\tilde{\omega}$ and σ are proportional to $1/(t_c-t)$ using constants $c_1=0.9$, $c_2=-0.9$, and $c_3=-0.99$ with initial values $b(0)=1$ and $c(0)=0.5$. The constant t_c is found to be 0.9495.

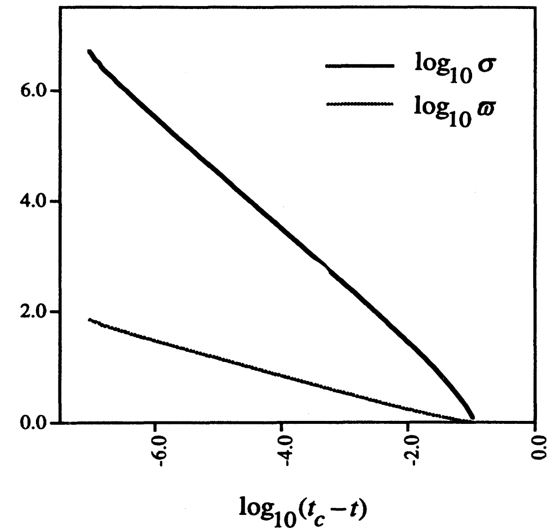

FIG. 2. Example of a singular solution of type (ii) where $\sigma=1/(t_c-t)$ and $\tilde{\omega}/\sigma\to 0$ using constants $c_1=10.1$, $c_2=-1.3$, and $c_3=-1.45$ with initial values $b(0)=1$ and $c(0)=10^{-6}$. The constant t_c is found to be 0.1153.

V. SINGULARITY DYNAMICS IN THE COMPLEX SPATIAL DOMAIN

As mentioned in Sec. I, Tanveer and Speziale [29] have proposed a method for investigating the singularity dynamics of the Euler equation in the complex physical domain. In this method, a class of smooth and bounded initial conditions in the real physical domain is analytically continued into the complex unphysical domain where there are complex singularities. Then the question of finite-time singularities in the real physical domain can be reduced to the question of whether the singularities in the complex domain reach the real domain in finite time.

The method proposed by Tanveer and Speziale [29] is based on some important assumptions that have not been shown to be valid in general. The method has been used successfully to obtain sufficient conditions for stability for a class of three-dimensional Euler flows identical to those found earlier by WKB methods [24,25]. In what follows, we suggest that some of the assumptions in Ref. [29] can be relaxed and that the method can be applied to our symmetric initial condition to obtain useful insights. There are open mathematical questions that we do not settle here, but since we state our assumptions clearly, it is our hope that the validity of the framework of Ref. [29] in the context of our model can be determined by further work.

For the purpose of this section, we consider $\mathbf{x}$ to be a complex variable and consider initial conditions for which the velocity and vorticity are real and analytic everywhere for real $\mathbf{x}$. However, the initial condition has complex singularities at the surface

$$d(\mathbf{x},0)=0\ , \tag{61}$$

where $d(\mathbf{x},0)$ is a real and positive analytic function for real $\mathbf{x}$. Following Ref. [29], we take the initial conditions for the velocity and the vorticity to be of the form

$$\mathbf{u}(\mathbf{x},0)=\mathbf{v}_s(\mathbf{x},0)+f[d(\mathbf{x},0)]\mathbf{q}(\mathbf{x},0) \tag{62}$$

and

$$\boldsymbol{\omega}(\mathbf{x},0)=\omega_s(\mathbf{x},0)+f'[d(\mathbf{x},0)]\mathbf{p}(\mathbf{x},0) , \tag{63}$$

respectively, where $f(d)$ is a function to be specified. Tanveer and Speziale assume that

$$f(d)=d^{\alpha}, \quad 0<\alpha<1 . \tag{64}$$

In (62) and (63), $\mathbf{v}_s(\mathbf{x},0),\omega_s(\mathbf{x},0),\mathbf{q}(\mathbf{x},0)$, and $\mathbf{p}(\mathbf{x},0)$ are real and analytic functions of $\mathbf{x}$ that obey the equations $\nabla\cdot\mathbf{v}_s(\mathbf{x},0)=0,\nabla\times\mathbf{v}_s(\mathbf{x},0)=\omega_s(\mathbf{x},0)$, and $\nabla\times[f\mathbf{q}(\mathbf{x},0)]=f'\mathbf{p}(\mathbf{x},0)$. It is claimed in Ref. [29] that for $t>0$, the complex solutions

$$\mathbf{u}(\mathbf{x},t)=\mathbf{v}_s(\mathbf{x},t)+f[d(\mathbf{x},t)]\mathbf{q}(\mathbf{x},t) , \tag{65}$$

$$\boldsymbol{\omega}(\mathbf{x},t)=\boldsymbol{\omega}_s(\mathbf{x},t)+f'[d(\mathbf{x},t)]\mathbf{p}(\mathbf{x},t) \tag{66}$$

may be constructed with

$$\nabla\cdot\mathbf{v}_s(\mathbf{x},t)=0 , \tag{67}$$

$$\nabla\times\mathbf{v}_s(\mathbf{x},t)=\boldsymbol{\omega}_s(\mathbf{x},t) , \tag{68}$$

$$\frac{\partial\boldsymbol{\omega}_s}{\partial t}+\mathbf{v}_s\cdot\nabla\boldsymbol{\omega}_s=\boldsymbol{\omega}_s\cdot\nabla\mathbf{v}_s , \tag{69}$$

with $\mathbf{p}$ and $\mathbf{q}$ satisfying Eqs. (21), (22), and (25) of Ref. [29]. The singular surface $d(\mathbf{x},t)=0$ is evolved according to the equation

$$\frac{\partial d(\mathbf{x},t)}{\partial t}+\mathbf{v}_s\cdot\nabla d(\mathbf{x},t)=0 . \tag{70}$$

Equation (70) implies that the singularities of $\boldsymbol{\omega}(\mathbf{x},t)$ for $t>0$ are determined by the relation $d(\mathbf{x},t)=0$. Thus Tanveer and Speziale obtain the surprising result that the location of the singularities in the complex domain can be determined from the smooth velocity field $\mathbf{v}_s(\mathbf{x},t)$ without the need to solve for $\mathbf{p}(\mathbf{x},t)$ and $\mathbf{q}(\mathbf{x},t)$, assuming that they exist. Tanveer and Speziale demonstrate that no complex singularity can reach the real domain in finite time if the variables $\mathbf{p},\mathbf{q},\mathbf{v}_s$, or $\boldsymbol{\omega}_s$ are smooth and develop no spontaneous singularities.

There has been some criticism [47] of Eq. (70) on the ground that it does not capture the nonlocal effect of pressure. We point out that whether that is so depends on the choice of $\mathbf{v}_s(\mathbf{x},0)$ and $\boldsymbol{\omega}_s(\mathbf{x},0)$, which influence the solutions of Eqs. (67) and (68) and hence the solution for $d(\mathbf{x},t)$. Furthermore, the nonlocal effect of pressure also enters into the equations for $\mathbf{p}$ and $\mathbf{q}$ that are assumed to exist.

A key difference between the considerations of Ref. [29] and what follows here is that we choose an exact but singular solution of the Euler equation for $\mathbf{v}_s$ and $\boldsymbol{\omega}_s$, i.e.,

$$\mathbf{v}_s=\frac{1}{t_c-t}\begin{bmatrix}-r/2\\ -\sqrt{3/2}r\\ z\end{bmatrix} , \tag{71}$$

$$\boldsymbol{\omega}_s=-\frac{\sqrt{3}}{t_c-t}\begin{bmatrix}0\\0\\1\end{bmatrix} . \tag{72}$$

Though motivated by our real analysis in Sec. III, the choice (71) does not rely in any way on the validity of that analysis. In order that $\mathbf{u}$ and $\boldsymbol{\omega}$ remain bounded initially for large $|\mathbf{x}|$, we assume that $f(d(\mathbf{x})\rightarrow\infty)\rightarrow 1$ and furthermore that

$$\mathbf{q}(\mathbf{x},0)=\mathbf{q}_1(\mathbf{x},0)-\mathbf{v}_s(\mathbf{x},0) , \tag{73}$$

where $\mathbf{q}_1(\mathbf{x},0)$ is smooth and decays to zero as $|\mathbf{x}|\rightarrow\infty$. We choose

$$f(d)=1-\exp(-d^{\alpha}), \quad 0<\alpha<1 . \tag{74}$$

Equation (74) gives $f(0)=0$, $f'(0)\rightarrow\infty$, which enables us to carry through with the approach of Ref. [29], assuming that there exist well-behaved solutions $\mathbf{p}$ and $\mathbf{q}$, satisfying Eqs. (21), (22), and (25) of Ref. [29], such that the complex singularities in (66) will not be canceled. We are not able to show definitely whether this assumption is true, because it involves proofs of existence of solutions to the cited partial differential equations of Ref. [29]. Further analytical and numerical work will be required to settle this issue.

The relation $d(\mathbf{x},t)=0$ is satisfied by complex points $\mathbf{x}(t)$, which obey the characteristic equation

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{v}_s(\mathbf{x}(t),t) , \tag{75}$$

subject to the initial condition (61). A solution of this equation is

$$d(\mathbf{x},t)=\frac{t_c}{t_c-t}\left[r^2+z^2\left[\frac{t_c-t}{t_c}\right]^3+b^2(0)\left[\frac{t_c-t}{t_c}\right]\right] \tag{76}$$

for $t<t_c$. We see that at $t=0$, complex singularities in vorticity occur on a cylindrical manifold, defined by

$$r^2+z^2+b^2(0)=0 . \tag{77}$$

For $t>0$, the coordinates of the movable singularities are determined by setting the right-hand side of Eq. (76) to zero. On the $z=0$ plane, we have

$$r(t)=ib(t)=ib(0)[(t_c-t)/t_c]^{1/2} . \tag{78}$$

Equation (78) suggests, remarkably, that not only do the complex singularities on the $z=0$ plane reach the origin (stagnation point) in finite time, but that they do so at a rate proportional to $(t_c-t)^{1/2}$. [From Eqs. (70) and (76) it might appear, upon first glance, that $d(\mathbf{x}=\mathbf{0},t)=\text{const}$ for all time. However, this would not be a correct inference from Eq. (76), which holds only for $t<t_c$. At $t=t_c$, the velocity $\mathbf{v}_s(\mathbf{x}=\mathbf{0},t)$ is no longer zero but becomes singular.] Here $b(t)$ defines a small space scale in the complex spatial domain that is distinct from the collapsing small scale of the flow $\mathbf{v}_s$. These results are consistent with the results obtained in Sec. III.

VI. REMARKS ON THE CONSTANTIN-FEFFERMAN THEOREM

Constantin [30,31] and Fefferman [31] have recently proved that "if the direction of vorticity is sufficiently well behaved in regions of high vorticity magnitude, then the solution is smooth." (The words "sufficiently well behaved" have a technical meaning that we consider carefully later.) This result imposes strong constraints on the possible singularities of the Euler equation. The crux of the theorem is in the relation [30]

$$\frac{d|\boldsymbol{\omega}|}{dt}=\alpha_C|\boldsymbol{\omega}| , \tag{79}$$

where

$$\alpha_C(\mathbf{x},t)=\frac{3}{4\pi}\mathrm{P}\int\frac{d\mathbf{y}}{|\mathbf{y}|^3}D(\hat{\mathbf{y}},\hat{\boldsymbol{\omega}}(\mathbf{x}+\mathbf{y},t),\hat{\boldsymbol{\omega}}(\mathbf{y},t)) , \tag{80}$$

$\hat{\boldsymbol{\omega}}\equiv\boldsymbol{\omega}/|\omega|$ is the unit vorticity vector and D is

$$D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)=(\mathbf{e}_1\cdot\mathbf{e}_3)[\mathrm{Det}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)] . \tag{81}$$

Since D vanishes if any of the column vectors in the determinant are parallel or antiparallel, spatial alignment or antialignment depletes the growth of vorticity, eliminating the possibility of a singularity. This is what happens in two dimensions globally. In three dimensions, in the neighborhood of a potential singularity, if D is too small, then it is not possible to support the singularity. This underscores the importance of infinite spatial gradients or small scales.

In view of the relation (79), the qualitative conclusions of the Constantin-Fefferman theorem are physically compelling. The question then is how our model measures up against this theorem. In Secs. III and V, we have established the growth of small scales. In particular, Contantin [30] shows that if the vorticity blows up self-similarly as

$$\omega(\mathbf{x},t)=\frac{1}{t_c-t}\mathbf{A}\left[\frac{\mathbf{x}}{a}\right]\equiv\frac{1}{t_c-t}\mathbf{A}(\mathbf{X}) , \tag{82}$$

where $a=\mu(t_c-t)^p$, then $p\geq 2/5$. This condition is satisfied by our locally self-similar solution. Furthermore, since the velocity blows up in the shrinking middle region, it is clear that our model also does not contradict a variant of the Constantin-Fefferman theorem, which states that there can be no singularity if the velocity does not blow up and the vorticity is locally absolutely integrable [31].

While accord with the theorem is reassuring, we discuss a technical issue that has to do with the words "sufficiently well behaved" in the first line of this section. In order to control the size of a "dangerous term" [31], the theorem needs the following assumption: There exist two positive constants Ω_C and ρ_C such that for every pair of two locations $|\mathbf{x}_1|$ and $|\mathbf{x}_2|$, whenever $|\boldsymbol{\omega}(\mathbf{x}_1,t)|>\Omega_C$ and $|\boldsymbol{\omega}(\mathbf{x}_2,t)|>\Omega_C$, we have

$$|\sin\phi(\mathbf{x}_1,\mathbf{x}_2,t)|\leq|\mathbf{x}_1-\mathbf{x}_2|/\rho_C , \tag{83}$$

where $\phi(\mathbf{x}_1,\mathbf{x}_2,t)$ is the angle between $\boldsymbol{\omega}(\mathbf{x}_1,t)$ and $\boldsymbol{\omega}(\mathbf{x}_2,t)$.

We now show that a self-similar vorticity field of the form (82) does not satisfy assumption (83). To see this, we choose two dimensionless vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ such that $\mathbf{X}_1\neq\mathbf{X}_2$, $\mathbf{A}(\mathbf{X}_1)||\mathbf{A}(\mathbf{X}_2)|\neq 0$, and $|\sin\phi|\neq 0$. We consider two corresponding locations in real space, $\mathbf{x}_1=a_m\mathbf{X}_1$ and $\mathbf{x}_2=a_m\mathbf{X}_2$, where $a_m=\mu(t_c-t_m)^p$. Then at $t=t_m$, we have

$$|\boldsymbol{\omega}(\mathbf{x}_i,t_m)|=\left[\frac{\mu}{a_m}\right]^{1/p}|\mathbf{A}(\mathbf{X}_i)| , \tag{84}$$

where $i=1,2$ and

$$\frac{|\sin\varphi(\mathbf{x}_1,\mathbf{x}_2,t_m)|}{|\mathbf{x}_1-\mathbf{x}_2|}=\frac{1}{a_m|\mathbf{X}_1-\mathbf{X}_2|}\left[1-\frac{|\mathbf{A}(\mathbf{X}_1)\cdot\mathbf{A}(\mathbf{X}_2)|^2}{|\mathbf{A}(\mathbf{X}_1)|^2|\mathbf{A}(\mathbf{X}_2)|^2}\right]^{1/2} . \tag{85}$$

From (85), we see that for every choice of Ω_C and ρ_C, we can find a small enough a_m such that $|\omega(\mathbf{x}_1,t)|>\Omega_C$ and $|\omega(\mathbf{x}_2,t)|>\Omega_C$, but $|\sin\phi(\mathbf{x}_1,\mathbf{x}_2,t_m)|>|\mathbf{x}_1-\mathbf{x}_2|/\rho_C$. We note that this holds even for the case $\mathbf{A}(\mathbf{X}=0)=0$, but $\mathbf{A}(\mathbf{X}\neq 0)\neq 0$. Thus assumption (83) is violated. Since the assumption of local self-similarity is inherent in our treatment, it is questionable whether the Constantin-Fefferman theorem is relevant to our model.

Before we conclude this section, we comment on case (ii), discussed in Sec. III. The argument given in the preceding paragraph enables us to see why the Beale-Kato-Majda theorem [44] is not violated in those cases where the velocity null coincides with a vorticity null, examples of which are discussed in Sec. III. From (82), it follows that $\max|\boldsymbol{\omega}(\mathbf{x},t)|=(t_c-t)^{-1}|\mathbf{A}(\mathbf{X}_m)|$ exists even when $\mathbf{A}(\mathbf{X}=0)=0$, but $\mathbf{A}(\mathbf{X}\neq 0)\neq 0$. Since $\mathbf{X}_m$ is independent of t, it follows that $\max|\boldsymbol{\omega}(\mathbf{x},t)|$ blows up as $(t_c-t)^{-1}$, consistent with the Beale-Kato-Majda constraint.

VII. EFFECT OF VISCOSITY

A. Connection with Lundgren's model

In Lundgren's model, a two-dimensional flow with vorticity $\omega_2(r,\theta,t)$ is placed in an axisymmetry straining flow with velocity components $u_z=s(t)z$ and $u_r=-s(t)r/2$, where $s(t)$ is the strain rate. This is shown to produce a three-dimensional axially strained flow with the vorticity

$$\omega_z(r,\theta,t)=S(t)\omega_2[\xi,\theta,T] , \tag{86}$$

where the variables S, ξ, and T are defined, respectively, by Eqs. (30), (31a), and (31d). At $t=0$, the vorticity of the three-dimensional flow is equal to the vorticity of the two-dimensional flow.

A connection can be established between the present model and Lundgren's by using Eqs. (27). If initially,

$$\frac{\partial}{\partial r_0}\sim\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\sim\frac{\partial}{\partial z_0} , \tag{87}$$

then as $t\rightarrow t_c$, we have

$$\frac{\partial}{\partial r}=\frac{\partial r_0}{\partial r}\frac{\partial}{\partial r_0}=\left[\frac{t_c}{t_c-t}\right]^{1/2}\frac{\partial}{\partial r_0}$$

$$\sim\frac{1}{r}\frac{\partial}{\partial\theta}=\left[\frac{t_c}{t_c-t}\right]^{1/2}\frac{1}{r_0}\frac{\partial}{\partial\theta_0}$$

$$\gg\frac{\partial}{\partial z}=\left[\frac{t_c-t}{t_c}\right]\frac{\partial}{\partial z_0}\ ,$$

which justifies setting $\partial/\partial z=0$.

We seek solutions of the Navier-Stokes equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\boldsymbol{\omega}=\boldsymbol{\omega}\cdot\nabla\mathbf{u}+\nu\nabla^2\boldsymbol{\omega}\ , \tag{88}$$

where ν is the viscosity. Near the origin where the vortex singularity occurs for inviscid flows, the vorticity has a dominant z component that is independent of z. We write

$$\omega_z=-\sqrt{3}s(t)+\alpha(r,\theta,t)\ . \tag{89}$$

The self-consistent velocity associated with this vorticity can be approximated as

$$u_z=s(t)z+\beta_z(r,\theta,t)\ , \tag{90}$$

$$u_r=-s(t)r/2+\beta_r(r,\theta,t)\ , \tag{91}$$

$$u_\theta=-(\sqrt{3}/2)s(t)r+\beta_\theta(r,\theta,t)\ . \tag{92}$$

Since $\nabla\cdot\mathbf{u}=0$, we get

$$\frac{\partial}{\partial r}(r\beta_r)+\frac{\partial\beta_\theta}{\partial\theta}=0\ , \tag{93}$$

which implies that β_r and β_θ are derivable from a stream function $\psi(r,\theta,t)$, i.e.,

$$\beta_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta},\quad \beta_\theta=-\frac{\partial\psi}{\partial r}\ . \tag{94}$$

From the z component of the equation $\boldsymbol{\omega}=\nabla\times\mathbf{u}$, we then get

$$\alpha=-\nabla^2\psi\ . \tag{95}$$

The dynamical equation for α follows from the z component of the Navier-Stokes equation (88)

$$\frac{\partial\alpha}{\partial t}+\left[\beta_r-\frac{sr}{2}\right]\frac{\partial\alpha}{\partial r}+\left[\frac{\beta_\theta}{r}-\frac{\sqrt{3}s}{2}\right]\frac{\partial\alpha}{\partial\theta}=\sqrt{3}\frac{\partial s}{\partial t}+s(-\sqrt{3}s+\alpha)+\nu\nabla^2\alpha\ . \tag{96}$$

As shown by Lundgren [32], by transforming to the variables defined by Eqs. (29)–(31), it is possible to construct solutions to Eq. (96) from solutions of the two-dimensional equation

$$\frac{\partial\alpha_2}{\partial T}+\frac{1}{\xi}\left[\frac{\partial\psi_2}{\partial\vartheta}\frac{\partial\alpha_2}{\partial\xi}-\frac{\partial\psi_2}{\partial\xi}\frac{\partial\alpha_2}{\partial\vartheta}\right]-\nu\nabla_\xi^2\alpha_2=-\frac{\sqrt{3}}{S^2}\left[s^2-\frac{\partial s}{\partial t}\right]\ , \tag{97}$$

where

$$\psi(r,\theta,t)=\psi_2(\xi,\vartheta,T)\ , \tag{98a}$$

$$\alpha(r,\theta,t)=S(t)\alpha_2(\xi,\vartheta,T)\ , \tag{98b}$$

$$\alpha_2(\xi,\vartheta,T)=-\nabla_\xi^2\psi_2\ , \tag{98c}$$

and

$$\nabla_\xi^2=\frac{\partial^2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{1}{\xi^2}\frac{\partial^2}{\partial\vartheta^2}\ . \tag{99}$$

For inviscid flows, we have shown earlier that $s(t)$ obeys Eq. (23), which yields a finite-time singularity of the vorticity. Then the stretch ratio $S(t)$ becomes infinitely large in finite time. (In contrast, the stretch ratio in Lundgren's model grows exponentially with time.) Due to this tendency for explosive growth, the right-hand side of Eq. (97) eventually becomes much smaller than the left-hand side and the equation reduces to

$$\frac{\partial\alpha_2}{\partial T}+\frac{1}{\xi}\left[\frac{\partial\psi_2}{\partial\vartheta}\frac{\partial\alpha_2}{\partial\xi}-\frac{\partial\psi_2}{\partial\xi}\frac{\partial\alpha_2}{\partial\vartheta}\right]-\nu\nabla_\xi^2\alpha_2=0\ . \tag{100}$$

The presence of even a small but finite viscosity thwarts the formation of the vortex singularity. This can be seen by introducing a viscous correction to Eq. (97). Using Eq. (21) to calculate the term $\nu\nabla^2\omega_z\cong4\sqrt{3}s\nu/a^2$, we can rewrite Eq. (23) as

$$\frac{\partial s}{\partial t}-s^2+\frac{4s\nu}{a^2}\cong0\ . \tag{101}$$

Thus, in the presence of viscosity, the strain saturates at a value s_c, which can be obtained by setting $\partial s/\partial t=0$. We then get the dissipation scale

$$a_d\cong2\left[\frac{\nu}{s_c}\right]^{1/2}\ . \tag{102}$$

The dissipation rate per unit mass (for the local flow) is

$$\varepsilon\equiv\frac{1}{2}\nu\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]^2\cong3\nu s_c^2\ . \tag{103}$$

Combining Eqs. (102) and (103), we get

$$a_d\cong2(3\nu^3/\varepsilon)^{1/4}\approx\eta\equiv(\nu^3/\varepsilon)^{1/4}\ , \tag{104}$$

where η is the Kolmogorov scale.

B. The spiral structure

It is clear by inspection of the initial conditions (Sec. II) that a spiral structure occurs naturally in the strained vortex solution. This spiral structure is also manifest in Eq. (28) as we go beyond the leading-order axisymmetric solution. We now follow Lundgren's method and seek two-dimensional vortex spiral solutions $\alpha_2(r,\theta,t)$ of the equation

$$\frac{\partial \alpha_2}{\partial t}+\frac{1}{r}\left[\frac{\partial \psi_2}{\partial \theta}\frac{\partial \alpha_2}{\partial r}-\frac{\partial \psi_2}{\partial r}\frac{\partial \alpha_2}{\partial \theta}\right] =\nu\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\alpha_2 . \quad (105)$$

Once the two-dimensional solution (i.e., the solution for which the axial strain $s=0$) to Eq. (105) is known, we can write down the three-dimensional axially strained solution by using Eq. (98b). For the two-dimensional equation (105), we seek solutions of the form

$$\alpha_2(r,\theta,t)=A_0(r,t)+\sum_{\pm} A_{\pm}(r,t)\exp\{\pm i3(\theta-\alpha t)\} , \quad (106)$$

where $\tilde{\alpha}=\beta_\theta/r$ is a slowly varying function of time. We write the stream function $\psi_2(r,\theta,t)$ as

$$\psi_2(r,\theta,t)=\Psi_0(r)+\Psi_1(r,\theta,t) , \quad (107)$$

where $\tilde{\alpha}\cong-(1/r^2)(d\Psi_0/dr)$. Hence Eq. (105) gives

$$\frac{\partial \alpha_2}{\partial t}+\tilde{\alpha}\frac{\partial \alpha_2}{\partial \theta}=\nu\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\alpha_2 , \quad (108)$$

where

$$A_0=\frac{1}{r}\frac{d}{dr}(r^2\tilde{\alpha}) . \quad (109)$$

The zeroth harmonic of the vorticity satisfies the heat equation

$$\frac{\partial A_0}{\partial t}=\nu\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]A_0 , \quad (110)$$

which has the solution $A_0=(1/t)\exp(-r^2/4\nu t)$. Hence A_0, as well as $\tilde{\alpha}$, has a slow decay time τ, defined by $\langle\tilde{\alpha}\rangle\tau=\langle\tilde{\alpha}\rangle a^2/\nu$, where $\langle\tilde{\alpha}\rangle$ is a characteristic magnitude of $\tilde{\alpha}$. As $\nu\to 0$, the harmonics $A_{\pm}$ obey the equation [32]

$$\frac{\partial A_{\pm}}{\partial t}\cong-9\nu\tilde{\alpha}'^2t^2A_{\pm} \quad (111)$$

and have the asymptotic behavior

$$A_{\pm}\cong f_{\pm}(r)\exp(-3\nu\tilde{\alpha}'^2t^3) , \quad (112)$$

where a prime means the derivative with respect to the argument and $f_{\pm}(r)$ represent arbitrary functions of r. Using Eq. (112), we can define a decay time for the harmonics by the relation

$$\langle\tilde{\alpha}\rangle\tau_{\pm}=(\langle\tilde{\alpha}\rangle a^2/\nu)^{1/3} . \quad (113)$$

Since the Reynolds number $\langle\tilde{\alpha}\rangle a^2/\nu$ is large, we have $\tau_{\pm}\ll\tau_0$, which means that the higher harmonics decay much faster than the zeroth harmonic. Due to nonlinear mode coupling, we expect that other harmonics will be generated though they are not present in the initial conditions. In what follows, we shall neglect the contribution of these higher harmonics.

Using the two-dimensional spiral solution in Eq. (98b), we obtain the axially strained spiral solution, given by

$$\alpha(r,\theta,t)=W_0(r,t)+\sum_{\pm} W_{\pm}(r,t)\exp\{\pm i3(\tilde{\theta}-\tilde{\alpha}T)\} , \quad (114)$$

where

$$W_0=SA_0(S^{1/2}r)=(S/T)\exp(-Sr^2/4\nu T) \quad (115)$$

and

$$W_{\pm}=Sf_{\pm}(S^{1/2}r)\exp[-3\{\tilde{\alpha}'(S^{1/2}r)\}^2\nu T^3] . \quad (116)$$

In Eqs. (115) and (116), the variables S and T will hereafter be evaluated using the strain rate $s=s_c$. Using Eq. (116), the fast decay time $T_{\pm}$ is seen to be

$$T_{\pm}=(3\nu\tilde{\alpha}'^2)^{-1/3} . \quad (117)$$

The rapidly oscillating factor $\exp(\mp i3\tilde{\alpha}T)\equiv\exp[\mp iR(r,t)]$ can be used to define a local wave number

$$q\equiv\partial R/\partial r=3S^{1/2}\tilde{\alpha}'T . \quad (118)$$

Using Eq. (117), q can be estimated to have the characteristic value

$$q_c\sim(s_c/\nu)^{1/2}\sim\eta^{-1} , \quad (119)$$

where η is defined by Eq. (101). Thus the Kolmogorov scale is an intrinsic feature of the spiral solution (including the zeroth and higher harmonics).

As discussed by Lundgren, Eq. (118) indicates that since S, as well as T, increases with time, the wave number q also increases with time while $\tilde{\alpha}'$ remains approximately constant. This mechanism of transfer of energy to small scales is associated with the tightening of the spiral structure, brought about by differential rotation and axial straining.

C. The Kolmogorov spectrum

As the vorticity intensifies to extremely large values, viscosity intervenes, thwarts the formation of the (inviscid) finite-time singularities, and causes transition to turbulence. It has been shown by Lundgren [32] that the energy spectrum of this system is given by

$$E(k)=\frac{C}{k^2}\int_0^{T_c}S^{1/2}(T)F_2(K,T)dT , \quad (120)$$

where $K\equiv kS^{-1/2}(T)$, T_c is the lifetime of a vortex segment of initial length l_0, and C is a constant defined as $C=2\pi^2l_0N_c/L^3$, where N_c is the rate of creation of vortices. (The parameter l_0N_c/L^3 represents the rate of creation of vortex length per unit volume.) The function F_2 is the enstrophy spectrum, defined by the equation,

$$F_2(k,t)=k\int_0^{2\pi}|\bar{\alpha}_2(k\cos\theta_k,k\sin\theta_k,t)|^2d\theta_k , \quad (121)$$

where

$$\bar{\alpha}_2(\mathbf{k},t)=(2\pi)^{-2}\int\int dr\,d\theta\,r\alpha_2(r,\theta,t)\exp(-i\mathbf{k}\cdot\mathbf{r}) . \quad (122)$$

Here **k** is defined on the two-dimensional (r,θ) plane and θ_k is the angle between the unit vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{x}}$.

The enstrophy spectrum (121) can be calculated by using the solutions of Eqs. (111) and (112). It has been shown [32–34] that the zeroth harmonic gives $E_0(k)\sim k^{-3}$. Since we anticipate that this will be subdominant to the contribution of the spirals to the energy spectrum for large k, we consider only the contribution of the spirals, described by the asymptotic solutions (112). Following the steps described in Ref. [32], we get

$$F_2(k,t)=\sum_{\pm} k\exp(-6\nu\tilde{\alpha}'^2t^3) \times \left|\int_0^\infty dr\, rf_\pm(r)J_{\pm 3}(kr)\exp(\mp 3i\tilde{\alpha}t)\right|^2 . \tag{123}$$

For large k and t, using the asymptotic expression for the Bessel function, we obtain a rapidly varying function in the integrand of the form $\exp\{-i(kr+3\tilde{\alpha}t)\}$. The integral in Eq. (123) can then be evaluated by the method of stationary phase. The result is

$$\left|\int_0^\infty dr\, rf_\pm(r)J_\pm(kr)\exp(\mp 3i\tilde{\alpha}t)\right|^2 \cong \frac{r_s|f_\pm(r_s)|^2}{2k\tilde{\alpha}''(r_s)t} , \tag{124}$$

where $r_s=r_s(k/t)$ is given by the stationary-phase condition $k+3\tilde{\alpha}'(r_s)t=0$. It is now apparent, by inspection of Eqs. (123) and (124), that the enstrophy spectrum $F_2(k,t)$ has the self-similar form $t^{-1}G(k/t)$ in the inertial range $L^{-1}\ll k\ll\eta^{-1}$. This similarity form for the enstrophy, which is preserved when dissipation can be neglected, is responsible for the Kolmogorov spectrum [32,34]. Inserting Eqs. (123) and (124) in Eq. (120), we obtain the contribution of the spirals to the energy spectrum

$$E_\pm(k)=C_K\varepsilon^{2/3}k^{-5/3}\exp(-\tfrac{2}{3}\eta^2k^2) , \tag{125}$$

with the coefficient

$$C_K=\frac{(\frac{2}{3})^{2/3}}{\Gamma(\frac{2}{3})}\frac{\varepsilon_\pm}{\varepsilon^{2/3}\nu^{1/3}s_c^{2/3}} . \tag{126}$$

Using the definitions

$$\xi_s\equiv S^{1/2}r_s, \quad \left|\frac{\tilde{\alpha}'}{\tilde{\alpha}''}\right|dT\equiv\frac{2}{3}Td\xi_s \tag{127}$$

and writing $s_c\approx S/T$, we obtain

$$\varepsilon_\pm=4\pi C\nu^{1/3}s_c\frac{\Gamma(\frac{2}{3})}{2^{2/3}}\left[\int_0^\infty d\xi_s\frac{\xi_s|f_\pm(\xi_s)|^2}{|\tilde{\alpha}'(\xi_s)|^{4/3}}\right] . \tag{128}$$

At first sight, C_K may appear to be model dependent, since Eq. (128) suggests so. However, since $E_\pm(k)\gg E_0(k)$ and $\varepsilon_\pm\approx\varepsilon$, we note that Eq. (126) reduces to

$$C_K\approx\frac{(\frac{2}{3})^{2/3}}{\Gamma(\frac{2}{3})}\left[\frac{\varepsilon}{\nu s_c^2}\right]^{1/3} . \tag{129}$$

If we recall that $\varepsilon\cong 3\nu s_c^2$, we obtain a constant $C_K\approx 0.8$.

VIII. SUMMARY AND DISCUSSION

In this paper, we have built upon a recent model of three-dimensional Euler flows that yields a finite-time vortex singularity [28]. In Ref. [28] as well as here, we have emphasized that *geometrical* features of an inviscid flow have a strong role in determining where unbounded local vorticity growth occurs in finite time. Our initial state, which contains two vorticity nulls with a velocity null in between, is of geometrical interest because nulls are a source of separatrices and singularities tend to occur near separatrices.

The finite-time vortex singularity in our model occurs at the stagnation point of the flow. The singularity develops at a point and is locally self-similar. The inviscid flow in the vicinity of the singular point has a complex spatial structure involving two disparate small scales.

There is a spiral structure in the initial conditions that is not essential to the formation of the vortex singularity. The presence of even a small but finite viscosity thwarts the formation of the vortex singularity and the spirals then provide a mechanism for energy cascade from the large to the small spatial scales, as originally envisioned by Lundgren [32]. Once the connection with Lundgren's model is established, the Kolmogorov spectrum follows after time averaging over the life of a vortex tube. As explained by Gilbert [34], the $k^{-5/3}$ spectrum is a robust consequence of time averaging that causes some remarkable cancellations in Lundgren's model.

An important question is how the results of this model can be connected to numerical (as well as real) experiments. There is an extensive database on vortex simulations that has grown out of careful numerical work over the past decade [5–9], including studies of vortex reconnection with antiparallel and orthogonal vortex tubes [48–54] in which no conclusive identification of vortex singularities have yet been made. There are, however, three recent numerical studies [45,46,55] in which finite-time vortex singularities growing as inverse time have been reported. We have made a qualitative connection with the singularities reported in one of them (Ref. [45]).

The model we have developed is quite simple and its assumptions need to be checked by further analytical and numerical studies. Yet the model seems to contain some essential attributes of turbulence. The qualitative picture that emerges from this paper begins with the (tendency of) unbounded local growth of vorticity in finite time at separatrices, defined by a network of nulls that act as attractors for the flow. This process leads to the growth of inviscid small scales. The vorticity intensification is eventually halted in the presence of even a small but finite viscosity. Spiral structures then provide a mechanism by which the Kolmogorov spectrum may be realized.

ACKNOWLEDGMENTS

We are pleased to acknowledge helpful discussions with R. Caflisch, J. Greene, and R. Pelz. We are indebted to S. Tanveer for prompting us to undertake the analysis in Sec. V and to him and S. Kida for constructive criticism of an earlier version of our work. This research was supported by the Air Force Office of Scientific Research Grant No. F49620-93-1-0071 and the National Science Foundation Grant No. ATM-9310157.

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